

A categorical action on quantized quiver varieties

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Abstract. In this paper, we describe a categorical action of any Kac-Moody algebra on a category of quantized coherent sheaves on Nakajima quiver varieties. By “quantized coherent sheaves,” we mean a category of sheaves of modules over a deformation quantization of the natural symplectic structure on quiver varieties. This action is a direct categorification of the geometric construction of universal enveloping algebras by Nakajima.

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Let \mathfrak{g} be an arbitrary Kac-Moody algebra with symmetric Cartan matrix, and Γ its associated Dynkin graph. Nakajima showed that there exists a remarkable connection between the algebra $U(\mathfrak{g})$ and certain varieties, called **quiver varieties**, constructed directly from the graph Γ . This construction takes the form of a map from $U(\mathfrak{g})$ to the Borel-Moore homology of a quiver analogue of the Steinberg variety [Nak98].

Both the source and target of this map have natural categorifications:

- the algebra $U(\mathfrak{g})$ is categorified by a 2-category \mathcal{U} . Actually several variations on the theme of this category have been introduced by Rouquier [Rou], Khovanov-Lauda [KL10], and Cautis-Lauda [CL]; we will use the formalism of the last of these. A 2-functor from this category into another 2-category is called a **categorical action** of \mathfrak{g} in this 2-category.
- the Borel-Moore homology of the “Steinberg” of a symplectic resolution \mathfrak{M} (such as a quiver variety) is categorified by a certain category of sheaves on $\mathfrak{M} \times \mathfrak{M}$. The structure sheaf of \mathfrak{M} possesses a quantization, in the sense

¹Supported by the NSF under Grant DMS-1151473 and by the NSA under Grant H98230-10-1-0199.

of [BK04], and the category of interest to us is that of bimodules over this quantization which satisfy a “Harish-Chandra” property, as described by Braden, Proudfoot and the author [BPW, §6.1-2]. Viewed correctly, these bimodules on the quiver varieties associated to a single highest weight λ can be organized into a 2-category, which we denote \mathcal{Q}^λ .

Thus, this previous work suggests how to categorify Nakajima’s map:

Theorem A *For each highest weight λ , there is a categorical representation of \mathfrak{g} in the 2-category \mathcal{Q}^λ ; taking “characteristic cycles” of these bimodules recovers the geometric construction of \mathcal{U} by Nakajima.*

Furthermore, the form of this functor is strongly suggested by Nakajima’s work; his map is defined by sending the Chevalley generators of $U(\mathfrak{g})$ to particular correspondences, called **Hecke correspondences**, which have natural moduli-theoretic significance. We “upgrade” these correspondences to modules over deformation quantizations, and show that these satisfy the categorical analogues of the Chevalley presentation.

We regard this theorem as very strong evidence of the naturality of the notion of categorical \mathfrak{g} -action currently circulating in the literature. While defined diagrammatically in a way that might outwardly seem arbitrary, in fact, its relations are hard-coded in the geometry of quiver varieties.

Very close analogues of the functors that appear in this representation have already been constructed in work of Zheng [Zhe] and Li [Lia, Lib]; however, these authors work in a slightly different context, which is based on constructible sheaves rather than deformation quantizations. The Riemann-Hilbert correspondence has already established a tie between constructible sheaves on a space X , and certain modules over a deformation quantization of T^*X , the differential operators on X . From this perspective, if there were a space Y of which a given Nakajima quiver variety were the cotangent bundle (there almost never is) then sheaves of modules over the quantized structure sheaf should be thought of as a replacement for the category of D-modules on Y . As pointed out by Zheng [Zhe, §2.2], his work was in a sense intended to understand constructible sheaves on Y .

Another perspective on these deformation quantizations is that they provide a replacement for the Fukaya category of a complex symplectic variety. Such a connection is suggested by Kapustin and Witten [KW07, §11] from a physical perspective, and the work of Nadler and Zaslow [NZ09] relating constructible sheaves and the Fukaya category of a cotangent bundle is also quite suggestive along these lines. In particular, it would be very interesting to find a categorical Lie algebra action in the 2-category of Lagrangian correspondences constructed by Wehrheim and Woodward [WW10]. Hopefully, instead of finding modules supported on the Hecke correspondences, one would simply consider them as objects in the Fukaya category.

Our main technical tool is a theorem of Cautis and Lauda [CL, Th. 1.1] which greatly reduces the number of relations which need to be checked in our to confirm that a candidate is a categorical action. In particular, the only actual relations between 2-morphisms we need to check have already been confirmed by Vasserot and Varagnolo [VV11]; the other conditions are either follow from general principles or are close analogues of results proven by Zheng and Li, with proofs that can be adapted easily.

Acknowledgements. This paper owes a great debt to Yiqiang Li; his work was an important inspiration, and he very helpfully pointed out a serious mistake in a draft version. I also want to thank Nick Proudfoot, Tony Licata and Tom Braden; I depended very much on previous work and conversations with them to be able to write this paper. I thank Sabin Cautis and Aaron Lauda for sharing an early version of their paper with me. I also appreciate very stimulating conversations with Catharina Stroppel, Ivan Losev and Peter Tingley.

Notation. We let Γ be an oriented graph and \mathfrak{g} the associated Kac-Moody algebra. Consider the weight lattice $Y(\mathfrak{g})$ and root lattice $X(\mathfrak{g})$, and the simple roots α_i and coroots α_i^\vee . Let $c_{ij} = \alpha_j^\vee(\alpha_i)$ be the entries of the Cartan matrix.

Choose an orientation Ω on Γ , let ϵ_{ij} denote the number of edges oriented from i to j , and fix

$$Q_{ij}(u, v) = (-1)^{\epsilon_{ij}}(u - v)^{c_{ij}}.$$

We let $U_q(\mathfrak{g})$ denote the deformed universal enveloping algebra of \mathfrak{g} ; that is, the associative $\mathbb{C}(q)$ -algebra given by generators E_i, F_i, K_μ for i and $\mu \in Y(\mathfrak{g})$, subject to the relations:

- i) $K_0 = 1, K_\mu K_{\mu'} = K_{\mu+\mu'}$ for all $\mu, \mu' \in Y(\mathfrak{g})$,
- ii) $K_\mu E_i = q^{\alpha_i^\vee(\mu)} E_i K_\mu$ for all $\mu \in Y(\mathfrak{g})$,
- iii) $K_\mu F_i = q^{\alpha_i^\vee(\mu)} F_i K_\mu$ for all $\mu \in Y(\mathfrak{g})$,
- iv) $E_i F_j - F_j E_i = \delta_{ij} \frac{\tilde{K}_i - \tilde{K}_{-i}}{q - q^{-1}}$, where $\tilde{K}_{\pm i} = K_{\pm d_i \alpha_i}$,
- v) For all $i \neq j$

$$\sum_{a+b=-c_{ij}+1} (-1)^a E_i^{(a)} E_j E_i^{(b)} = 0 \quad \text{and} \quad \sum_{a+b=-c_{ij}+1} (-1)^a F_i^{(a)} F_j F_i^{(b)} = 0.$$

1. THE 2-CATEGORY \mathcal{U}

Our primary object of study is a 2-category categorifying the universal enveloping algebra; versions of this category have been considered by Rouquier [Rou], Khovanov and Lauda [KL10] and Cautis and Lauda [CL]. It is most convenient for us to consider the precise construction given by the last. Since the construction of these categories is rather complex, we give a somewhat abbreviated description.

We let $t_{ij} = Q_{ij}(1, 0) = (-1)^{\epsilon_{ij}}$. By convention $t_{ii} = 1$. (We should warn the reader that in [CL] this scalar is allowed to be any non-zero number; we avoided this in order to simplify our relations). We define a category \mathcal{U}' where

- an object of this category is a weight $\lambda \in Y$.
- a 1-morphism $\lambda \rightarrow \mu$ is a formal sum of words in the symbols \mathcal{E}_i and \mathcal{F}_i where i ranges over Γ of weight $\lambda - \mu$, \mathcal{E}_i and \mathcal{F}_i having weights $\pm\alpha_i$. In [Rou], the corresponding 1-morphisms are denoted E_i, F_i , but we use these for elements of $U_q(\mathfrak{g})$. Composition is simply concatenation of words. In fact, we will take idempotent completion, and thus add a new 1-morphism for every projection from a 1-morphism to itself (once we have added 2-morphisms).

By convention, $\mathcal{F}_{\mathbf{i}} = \mathcal{F}_{i_n} \cdots \mathcal{F}_{i_1}$ if $\mathbf{i} = (i_1, \dots, i_n)$. We should warn the reader, this convention requires us to read our diagrams differently from the conventions of [Lau10, KL10, CL]; in our diagrammatic calculus, 1-morphisms point from the left to the right, not from the right to the left as indicated in [Lau10, §4].

- 2-morphisms are a quotient of the \mathbb{k} -span of certain immersed oriented 1-manifolds carrying an arbitrary number of dots whose boundary is given by the domain sequence on the line $y = 1$ and the target sequence on $y = 0$. We require that any component begin and end at like-colored elements of the 2 sequences, and that they be oriented upward at an \mathcal{E}_i and downward at an \mathcal{F}_i . We will describe their relations momentarily. We require that these 1-manifolds satisfy the same genericity assumptions as projections of tangles (no triple points or tangencies), but intersections are not over- or under-crossings; our diagrams are genuinely planar. We consider these up to isotopy which preserves this genericity.

We draw these 2-morphisms in the style of Khovanov-Lauda, by labeling the regions of the plane by the weights (objects) that the 1-morphisms are acting on.

By Morse theory, we can see that these are generated by

- * a cup $\epsilon : \mathcal{E}_i \mathcal{F}_i \rightarrow \emptyset$ or $\epsilon' : \mathcal{F}_i \mathcal{E}_i \rightarrow \emptyset$ and a cap $\iota' : \emptyset \rightarrow \mathcal{E}_i \mathcal{F}_i$ or $\iota : \emptyset \rightarrow \mathcal{F}_i \mathcal{E}_i$

$$\begin{array}{cccc} \epsilon = \begin{array}{c} \lambda \\ i \quad \quad i \\ \curvearrowright \\ \lambda + \alpha_i \end{array} & \epsilon' = \begin{array}{c} \lambda \\ i \quad \quad i \\ \curvearrowleft \\ \lambda - \alpha_i \end{array} & \iota' = \begin{array}{c} \lambda \\ \curvearrowright \\ i \quad \quad i \\ \lambda - \alpha_i \end{array} & \iota = \begin{array}{c} \lambda \\ \curvearrowleft \\ i \quad \quad i \\ \lambda + \alpha_i \end{array} \end{array}$$

- * a crossing $\psi : \mathcal{F}_i \mathcal{F}_j \rightarrow \mathcal{F}_j \mathcal{F}_i$ and a dot $y : \mathcal{F}_i \rightarrow \mathcal{F}_i$

$$\begin{array}{cc} \psi = \begin{array}{c} \lambda \\ j \quad \quad i \\ \diagdown \quad \diagup \\ i \quad \quad j \\ \lambda \end{array} & y = \begin{array}{c} \lambda \\ i \\ \bullet \\ i \\ \lambda \end{array} \end{array}$$

Before writing the relations, let us remind the reader that these 2-morphism spaces are actually graded; the degrees are given by

$$\begin{aligned} \deg \begin{array}{c} \diagup \diagdown \\ i \quad j \end{array} &= -\langle \alpha_i, \alpha_j \rangle & \deg \begin{array}{c} \downarrow \\ i \end{array} &= \langle \alpha_i, \alpha_i \rangle & \deg \begin{array}{c} \diagdown \diagup \\ i \quad j \end{array} &= -\langle \alpha_i, \alpha_j \rangle & \deg \begin{array}{c} \uparrow \\ i \end{array} &= \langle \alpha_i, \alpha_i \rangle \\ \\ \deg \begin{array}{c} i \quad \lambda \\ \frown \end{array} &= -\langle \lambda, \alpha_i \rangle - d_i & \deg \begin{array}{c} i \quad \lambda \\ \smile \end{array} &= \langle \lambda, \alpha_i \rangle - d_i \\ \deg \begin{array}{c} i \quad \lambda \\ \smile \end{array} &= -\langle \lambda, \alpha_i \rangle - d_i & \deg \begin{array}{c} i \quad \lambda \\ \frown \end{array} &= \langle \lambda, \alpha_i \rangle - d_i. \end{aligned}$$

The relations satisfied by the 2-morphisms include:

- the cups and caps are the units and counits of a biadjunction. The morphism y is cyclic, whereas the morphism ψ is double right dual to $t_{ij}/t_{ji} \cdot \psi$ (see [CL] for more details).
- Any bubble of negative degree is zero, any bubble of degree 0 is equal to 1. We must add formal symbols called “fake bubbles” which are bubbles labelled with a negative number of dots (these are explained in [KL10, §3.1.1]); given these, we have the inversion formula for bubbles, shown in Figure 1.

$$\sum_{k=\lambda^i-1}^{j+\lambda^i+1} \begin{array}{c} \text{bubble with } k \text{ dots} \\ \lambda \end{array} = \begin{cases} 1 & j = -2 \\ 0 & j > -2 \end{cases}$$

FIGURE 1. Bubble inversion relations; all strands are colored with α_i .

- 2 relations connecting the crossing with cups and caps, shown in Figure 2.
- Oppositely oriented crossings of differently colored strands simply cancel, shown in Figure 3.
- the endomorphisms of words only using \mathcal{F}_i (or by duality only \mathcal{E}_i 's) satisfy the relations of the **quiver Hecke algebra** R , shown in Figure 4.

As in [KL10], we let \mathcal{U} denote the 2-category where every Hom-category is replaced by its idempotent completion; we note that since every object in \mathcal{U} has a finite-dimensional degree 0 part of its endomorphism algebra, every Hom-category satisfies the Krull-Schmidt property.

This 2-category is a categorification of the universal enveloping algebra in the sense that

Theorem 1.1 ([Web, 1.7-9]) *The Grothendieck group of \mathcal{U} is isomorphic to $\dot{\mathcal{U}}$.*

This theorem was first conjectured by Khovanov and Lauda [KL10] and proven by them in the special case of \mathfrak{sl}_n .

$$\begin{aligned}
 \lambda \text{ (cap)} &= - \sum_{a+b=-1} \text{ (bubble with } a \text{ and } b \text{ dots)} \lambda \\
 \lambda \text{ (cup)} &= \sum_{a+b=-1} \text{ (bubble with } a \text{ and } b \text{ dots)} \lambda \\
 \lambda \text{ (cross)} &= \lambda \text{ (vertical lines)} + \sum_{a+b+c=-1} \text{ (bubble with } a, b, c \text{ dots)} \lambda \\
 \lambda \text{ (cross)} &= \lambda \text{ (vertical lines)} + \sum_{a+b+c=-1} \text{ (bubble with } a, b, c \text{ dots)} \lambda
 \end{aligned}$$

FIGURE 2. “Cross and cap” relations; all strands are colored with α_i . By convention, a negative number of dots on a strand which is not closed into a bubble is 0.

$$\begin{aligned}
 \lambda \text{ (cross, } i \text{ over } j) &= t_{ij} \text{ (vertical lines, } i \text{ over } j) \lambda \\
 \lambda \text{ (cross, } j \text{ over } i) &= t_{ji} \text{ (vertical lines, } j \text{ over } i) \lambda
 \end{aligned}$$

FIGURE 3. The cancellation of oppositely oriented crossings with different labels.

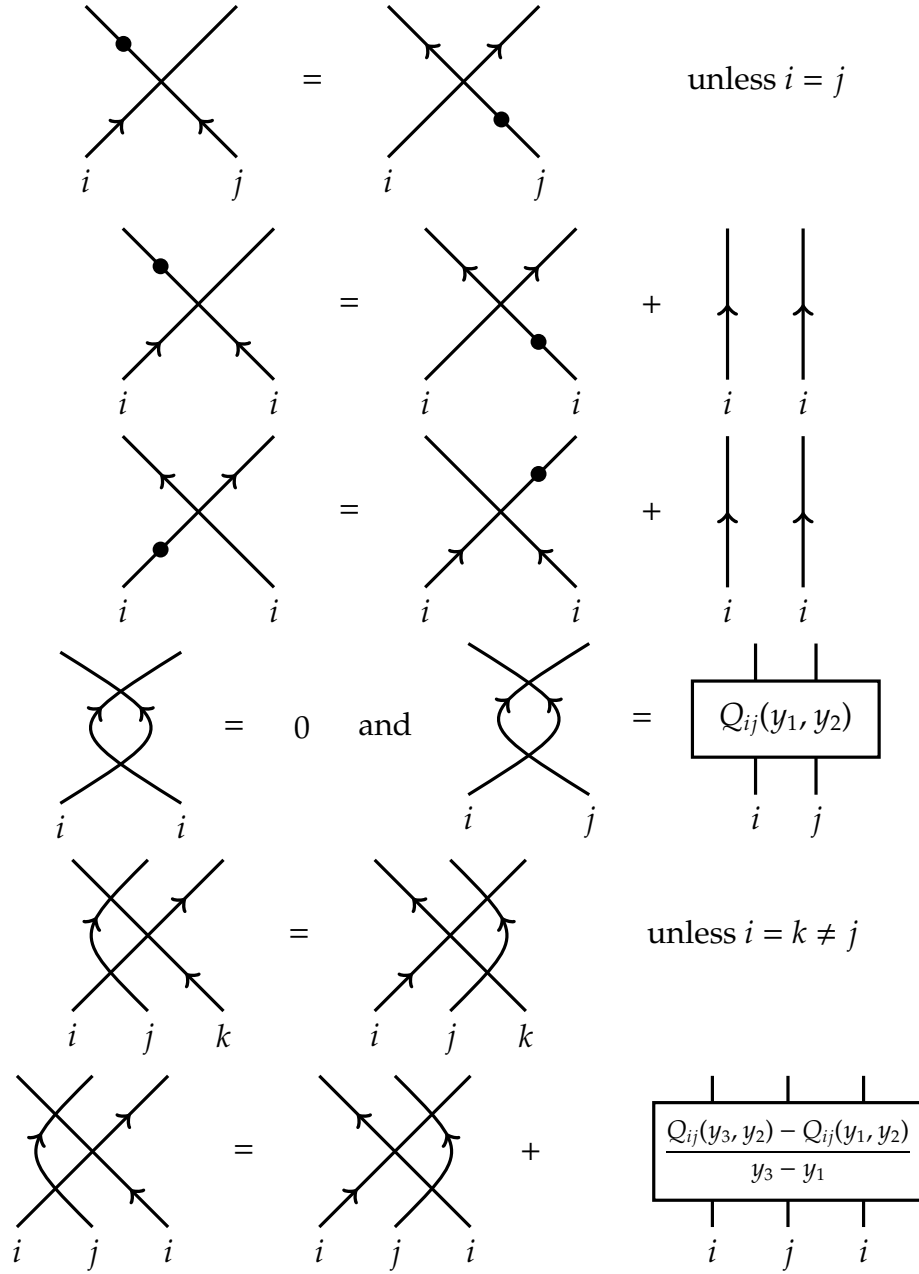


FIGURE 4. The relations of the quiver Hecke algebra. These relations are insensitive to labeling of the plane.

2. QUIVER VARIETIES

Recall that Γ denotes the Dynkin graph of \mathfrak{g} .

Definition 2.1 For each orientation Ω of Γ (thought of as a subset of the edges of the oriented double), a **representation of (Γ, Ω) with shadows** is

- a pair of finite dimensional \mathbb{C} -vector spaces V and W , graded by the vertices of Γ , and
- a map $x_e : V_{\omega(e)} \rightarrow V_{\alpha(e)}$ for each oriented edge (as usual, α and ω denote the head and tail of an oriented edge), and
- a map $z : V \rightarrow W$ that preserves grading.

We let \mathbf{w} and \mathbf{v} denote Γ -tuples of integers.

For now, we fix an orientation Ω , though we will sometimes wish to consider the collection of all orientations. With this choice, we have the **universal (\mathbf{w}, \mathbf{v}) -dimensional representation**

$$E_{\mathbf{v}, \mathbf{w}} = \bigoplus_{i \rightarrow j} \text{Hom}(\mathbb{C}^{v_i}, \mathbb{C}^{v_j}) \oplus \bigoplus_i \text{Hom}(\mathbb{C}^{v_i}, \mathbb{C}^{w_i}).$$

In moduli terms, this is the moduli space of actions of the quiver (in the sense above) on the vector spaces $\mathbb{C}^{\mathbf{v}}, \mathbb{C}^{\mathbf{w}}$, with their chosen bases considered as additional structure.

If we wish to consider the moduli space of representations where V has fixed graded dimension (rather than of actions on a fixed vector space), we should quotient by the group of isomorphisms of quiver representations: $G_{\mathbf{v}} = \prod_i \text{GL}(\mathbb{C}^{v_i})$ acting by pre- and post-composition. The result is the **moduli stack of \mathbf{v} -dimensional representations shadowed by $\mathbb{C}^{\mathbf{w}}$** , which we can define as the stack quotient

$$X_{\mathbf{v}}^{\mathbf{w}} = E_{\mathbf{v}, \mathbf{w}} / G_{\mathbf{v}}.$$

This is not a scheme in the usual sense, but rather a smooth Artin stack. Those readers made skittish by the mention of stacks can consider this as a purely formal symbol whose derived category is the equivariant derived category of $E_{\mathbf{v}, \mathbf{w}}$ in whatever sense they like, for example, as in the book of Bernstein and Lunts [BL94] or as described by the author and Williamson [WW]. We will always consider this space as having the classical topology.

By convention, if $w_i = \alpha_i^\vee(\lambda)$ and $\mu = \lambda - \sum v_i \alpha_i$, then $X_{\mu}^{\lambda} = X_{\mathbf{v}}^{\mathbf{w}}$ (if the difference is not in the positive cone of the root lattice, then this is by definition empty), and $X^{\lambda} = \dot{\sqcup}_{\xi} X_{\xi}^{\lambda}$.

The geometric construction builds on the work of Li [Lia] and that of Zheng [Zhe]. Li defines a 2-category built from perverse sheaves on double framed quiver varieties.

Definition 2.2 *Li's 2-category is defined as follows:*

- 0-morphisms are dimension vectors for the quiver Γ ,
- 1-morphisms between \mathbf{d} and \mathbf{d}' are objects of geometric origin in the localized derived category which Li denotes by $\mathcal{D}^-(E_{\Omega}(k^{\lambda}, k^{\mathbf{d}}, k^{\mathbf{d}'}))$, with product given by the convolution product of [Lia, (16)].
- 2-morphisms are morphisms in the category described above.

For certain technical purposes, it is much more convenient for us to use a different 2-category built using quantizations. Let

$$\mathfrak{M}_\mu^\lambda = T^*E_\mu^\lambda //_{\det} G_\mu = \mu^{-1}(0)^s / G_\mu$$

be the Nakajima quiver variety attached to λ and μ ; this is a smooth, quasi-projective variety which arises through geometric invariant theory as an open subset of the cotangent bundle of X_μ^λ . Any point in $T^*E_\mu^\lambda$ can be thought of as a representation of the doubled quiver of Γ (with the framing maps also doubled). The subset $\mu^{-1}(0)$ can be thought of as parameterizing representations that descend to a certain quotient of the doubled path algebra called the **preprojective algebra**. In this language $\mu^{-1}(0)^s$ is the subset where no subrepresentation killed by all shadow maps exists. See [Nak94, Nak98] for a more detailed discussion of the geometry of these varieties.

By completely general techniques, we can construct a sheaf of algebras \mathcal{A}'_μ on \mathfrak{M}_μ^λ which “quantizes” the structure sheaf, in the sense that \mathcal{A}'_μ is a sheaf of free $\mathbb{C}[[h]]$ -algebras with $\mathcal{A}'_\mu / h\mathcal{A}'_\mu \cong \mathcal{O}_{\mathfrak{M}_\mu^\lambda}$. Such quantizations are discussed in general in [BK04], and in the context of symplectic resolutions of singularities such as quiver varieties in [BPW]. For such a quantization, we let $\mathcal{A}_\mu = \mathcal{A}'_\mu[[h^{-1}]]$.

As is shown in [BK04], such a quantization is not unique, but rather is determined by its period, a power series in $hH^2(\mathfrak{M}; \mathbb{C}[[h]])$. One obvious choice of quantization is that corresponding to the class 0, the canonical quantization. This is **not** the choice we take.

Instead, by [KR08, 2.8(i)], this abstractly constructed quantization can actually be constructed by Hamiltonian reduction of differential operators on E_μ^λ by a quantized moment map sending an element of the Lie algebra \mathfrak{g}_μ to the vector field giving its action of E_μ^λ plus a some character χ .

Definition 2.3 *We let \mathcal{A}'_μ be the sheaf given by the reduction by obvious moment map for \mathfrak{g}_μ acting on functions ($\chi = 0$) and let $\mathcal{A}_\mu = \mathcal{A}'_\mu[[h^{-1}]]$.*

On X_ν^w , we have a tautological vector bundle \mathcal{V}_i whose fiber over a representation is the part of that representation at node i ; let $\mathcal{L}_i = \det(\mathcal{V}_i)$. By [BPW, 6.4], the period of \mathcal{A}'_μ is

$$\frac{1}{2} \sum_{i \in \Gamma} \left(w_i + \sum_{j \rightarrow i} v_j - \sum_{i \rightarrow j} v_j \right) c_1(\mathcal{L}_i) h.$$

Note that this period depends on the choice of orientation of Γ , but its class modulo $hH^2(\mathfrak{M}_\mu^\lambda; \mathbb{Z})$ does not. Also, this is not always an integral class; this is a generalization of the fact that differential operators, thought of as a quantization of a cotangent bundle, do not always have integral period (as [BPW, 3.10] shows). If, as suggested in the introduction, we think of the quiver variety as the cotangent bundle of a hypothetical space Y , this would be the algebra of untwisted differential operators

on Y , and the quantization with period 0 would be the differential operators in the square root of the canonical bundle of Y .

The quiver varieties carry a natural \mathbb{C}^* -action inherited from the fiber scaling action on $T^*E_{\mathbf{v},\mathbf{w}}$. The sheaf of algebras \mathcal{A}_μ carries a equivariant structure over \mathbb{C}^* (see [Los12, 2.3.3]). We let $\mathcal{A}_\mu\text{-mod}$ denote the category of \mathbb{C}^* -equivariant good modules over \mathcal{A}_μ (as defined in [BPW, §4]).

The Hamiltonian reduction realization of \mathcal{A}_μ gives us a functor $r : \mathcal{D}_{X_\mu^\lambda}\text{-mod} \rightarrow \mathcal{A}_\mu\text{-mod}$ (called the “Kirwan functor” in [BPW]) from D-modules on X_μ^λ to \mathcal{A}_μ -modules; if we used a different quantized moment map, we would instead have functors from representation categories of various sheaves of twisted differential operators on X_μ^λ .

An account of the general framework for this construction is given by Braden, Proudfoot and the author in [BPW, §5.4]. In particular, this functor admits a left adjoint $r_! : \mathcal{A}_\mu\text{-mod} \rightarrow \mathcal{D}_{X_\mu^\lambda}\text{-mod}$, such that $r \circ r_! \cong \text{id}$ by [BPW, 5.17]. Furthermore, as explained in [BPW, 5.18], there is a right adjoint r_* which may not preserve coherence (though it often does); instead, r_* lands in the ind-category $\widehat{\mathcal{D}_{X_\mu^\lambda}\text{-mod}}$ of $\mathcal{D}_{X_\mu^\lambda}\text{-mod}$, the category of all countable direct limits in this category. Still, this adjoint also satisfies $r \circ r_* \cong \text{id}$.

We let \mathcal{Q}^λ be the 2-category where

- 0-morphisms are dimension vectors for the quiver Γ ,
- 1-morphisms between \mathbf{d} and \mathbf{d}' given by the bounded-below derived category of complexes of modules over $\mathcal{A}_\mu \boxtimes \mathcal{A}_{\mu'}^{op}$. Composition of 1-morphisms $\mathcal{H}_1 : \mathfrak{M}_{\mu_1}^\lambda \rightarrow \mathfrak{M}_{\mu_2}^\lambda$ and $\mathcal{H}_2 : \mathfrak{M}_{\mu_2}^\lambda \rightarrow \mathfrak{M}_{\mu_3}^\lambda$ is given by convolution

$$(1) \quad \mathcal{H}_1 \star \mathcal{H}_2 := (p_{13})_*(p_{12}^* \mathcal{H}_1 \otimes_{p_{23}^*}^L \mathcal{H}_2)[- \dim(\mathfrak{M}_{\mu_1}^\lambda \times \mathfrak{M}_{\mu_3}^\lambda)].$$

- 2-morphisms are morphisms in the category described above; we consider this as a graded category with the homological grading.

This 2-category receives a natural 2-functor from the analytic version of Li’s 2-category; the (classical topology) derived category of X_μ^λ has a functor to the derived category of $\mathcal{D}_{X_\mu^\lambda}$ -modules given by the Riemann-Hilbert correspondence, and the functor r kills the necessary subcategories to induce a 2-functor from the localization. In order to confirm that this is a 2-functor, we check that we could also define convolution as in Li’s category [Lia, §4.8] (though, his definition is “dual” to ours, since he uses the left, rather than right adjoint of reduction).

3. HECKE CORRESPONDENCES AND CATEGORICAL ACTIONS

We let $X_{\xi,\nu}^\lambda$ denote the moduli stack of short exact sequences (“Hecke correspondences”) where the subobject belongs in X_ξ^λ , the total object in $X_{\xi-\nu}^\lambda$ and the quotient in $X_{-\nu}^0$. This is a quotient of the variety of pairs consisting of a point in $E_{\mathbf{w},\mathbf{v}}$, and an

invariant collection of subspaces of k^{v_i} of codimension $\omega_i^\vee(v)$ by the natural action of the group G_v . Thus, this moduli stack is equipped with projections

$$\begin{array}{ccccc} & & X_{\xi,v}^\lambda & & \\ & p_1 \swarrow & \downarrow p_2 & \searrow p_3 & \\ X_\xi^\lambda & & X_{\xi-v}^\lambda & & X_{-v}^0 \end{array}$$

For each λ, μ, i , we let

$$\tilde{\mathcal{F}}_i = \omega_{X_{\mu-\alpha_i}^\lambda} \otimes_{\mathcal{O}_{X_{\mu-\alpha_i}^\lambda}} (p_1 \times p_2)_* \mathcal{O}_{X_{\mu\alpha_i}^\lambda} \quad \mathcal{F}_i = r(\tilde{\mathcal{F}}_i)$$

$$\tilde{\mathcal{E}}_i = \omega_{X_\mu^\lambda} \otimes_{\mathcal{O}_{X_\mu^\lambda}} (p_2 \times p_1)_* \mathcal{O}_{X_{\mu\alpha_i}^\lambda} \quad \mathcal{E}_i = r(\tilde{\mathcal{E}}_i)$$

Naturally, \mathcal{F}_i is a module over $\mathcal{A}_\mu \boxtimes \mathcal{A}_{\mu-\alpha_i}^{op}$ and \mathcal{E}_i is a module over $\mathcal{A}_{\mu-\alpha_i} \boxtimes \mathcal{A}_\mu^{op}$. That is, by definition, these are 1-morphisms in \mathcal{Q}^λ between the appropriate dimension vectors. They are the images under the Riemann-Hilbert correspondence of the similarly named objects in Li's development of the theory. We now proceed to our principle result:

Theorem 3.1 *We have a 2-functor of graded categories $\mathcal{G}_\lambda: \mathcal{U} \rightarrow \mathcal{Q}^\lambda$ sending $\mathcal{E}_i \mapsto \mathcal{E}_i$ and $\mathcal{F}_i \mapsto \mathcal{F}_i$.*

For now, we postpone the proof of this theorem to Section 4, and instead discuss its variations and consequences in a bit more detail.

The existence of the objects \mathcal{E}_i and \mathcal{F}_i depends very strongly on the fact that we use untwisted D-modules here. Consider twists

$$\chi_1 = \sum_{j \in \Gamma} a_j c_1(\mathcal{L}_j) \in H^2(X_\mu^\lambda) \quad \chi_2 = \sum_{j \in \Gamma} b_j c_1(\mathcal{L}'_j) \in H^2(X_{\mu-\alpha_i}^\lambda)$$

on the respective varieties, where we use $\mathcal{L}'_j, \mathcal{V}'_j$ to denote the tautological bundles on $X_{\mu-\alpha_i}^\lambda$.

Proposition 3.2 *There exists a line bundle \mathcal{L} on $X_{\mu;\alpha_i}^\lambda$ such that*

$$\tilde{\mathcal{F}}_i^\mathcal{L} = \omega_{X_{\mu-\alpha_i}^\lambda} \otimes_{\mathcal{O}_{X_{\mu-\alpha_i}^\lambda}} (p_1 \times p_2)_* \mathcal{L}$$

$$\tilde{\mathcal{E}}_i^\mathcal{L} = \omega_{X_\mu^\lambda} \otimes_{\mathcal{O}_{X_\mu^\lambda}} (p_2 \times p_1)_* \mathcal{L}^{-1}$$

are bimodules over $\mathcal{D}_{X_\mu^\lambda}(\chi_1)$ and $\mathcal{D}_{X_{\mu-\alpha_i}^\lambda}(\chi_2)$ if and only if $a_i, b_i, a_j - b_j \in \mathbb{Z}$ for all $j \in \Gamma$.

Proof. First we note that for $X_\mu^\lambda, X_{\mu-\alpha_i}^\lambda$ and $X_{\mu;\alpha_i}^\lambda$, the Picard group is just $H^2(-; \mathbb{Z})$, and these are identified with the character group of the groups G_μ and $G_{\mu-\alpha_i}$ and a maximal parabolic in the latter, respectively. In practice, this means that

- $\{c_1(\mathcal{L}_j)\}_{j \in \Gamma}$ is a basis of $H^2(X_\mu^\lambda; \mathbb{Z})$,
- $\{c_1(\mathcal{L}'_j)\}_{j \in \Gamma}$ is a basis of $H^2(X_{\mu-\alpha_i}^\lambda; \mathbb{Z})$ and
- $\{c_1(p_1^* \mathcal{L}_j)\}_{j \in \Gamma} \cup \{c_1(p_2^* \mathcal{L}'_i)\}$ for $H^2(X_{\mu; \alpha_i}^\lambda; \mathbb{Z})$.

In order to have the twisted D-module actions, we desire, we must have $c_1(\mathcal{L}) = p_1^* \chi_1 - p_2^* \chi_2$; by the identification of the Picard group by homology, such an \mathcal{L} exists if and only if $p_1^* \chi_1 - p_2^* \chi_2 \in H^2(X_{\mu; \alpha_i}^\lambda; \mathbb{Z})$.

For $j \neq i$, we have that $p_1^* c_1(\mathcal{L}_j) = p_2^* c_1(\mathcal{L}'_j)$, but for i , these are independent classes. Thus, we have that

$$p_1^* \chi_1 - p_2^* \chi_2 = a_i p_1^* c_1(\mathcal{L}_i) - b_i p_2^* c_1(\mathcal{L}'_i) + \sum_{j \neq i} (a_j - b_j) p_1^* c_1(\mathcal{L}_j),$$

which is integral if and only if $a_i, b_i, a_j - b_j \in \mathbb{Z}$. \square

Thus more generally, using Proposition 3.2, we can define such an action where we choose any quantization corresponding to differential operators in a line bundle on each X_μ^λ , not just the particular one we have fixed. If we instead choose a not necessarily integral twist $\chi = \sum a_i c_1(\mathcal{L}_i)$, we only know at the moment how to construct a categorical action of the smaller Lie algebra generated by the simple root spaces where a_i is integral.

This observation is particularly interesting in the case where \mathfrak{g} is affine, λ is the basic fundamental weight and $\mu = n\delta$. In this case, the \mathbb{C}^* -invariant section algebra $\Gamma(\mathfrak{M}_\mu^\lambda; \mathcal{A}_\mu)^{\mathbb{C}^*}$ is a spherical symplectic reflection algebra for S_n wr γ , where γ matches \mathfrak{g} under the Mackay correspondence by [EGGO07, Gor06, Los12]; this phenomenon of functors associated to roots appearing when particular functions on the parameter space are integral is quite suggestive in connection with Etingof's conjecture relating finite dimensional modules for these symplectic reflection algebras to affine Lie algebras [Eti]. However, the form of these conjectures suggest that there are functors associated to all roots where $\sum_i a_i \omega_i^\vee(\alpha) \in \mathbb{Z}$. Bezrukavnikov and Losev are currently investigating such functors at the moment, taking the approach of defining them using a categorical lift of the Weyl group action [BL].

The 2-functor \mathcal{G}_λ actually lands in a much smaller subcategory of \mathcal{Q}^λ . In the product $\mathfrak{M}_\mu^\lambda \times \mathfrak{M}_{\mu'}^\lambda$, we still have a notion of “diagonal;” the affinization of a quiver variety is the moduli space of semi-simple representations of the pre-projective algebra of a given dimension. We say a pair of such representations lies in the **stable diagonal** if they become isomorphic after the addition of trivial representations (that is, they are isomorphic up to stabilization).

Following Nakajima, we let Z denote the preimage of the stable diagonal. In [BPW, §6.1], Braden, Proudfoot and the author define a 2-subcategory $\mathbf{HC}^g(\lambda)$ of good sheaves of $\mathcal{A}_\mu \boxtimes \mathcal{A}_{\mu'}$ -modules called **Harish-Chandra bimodules**. This is the category of modules \mathcal{M} such that:

- the support of \mathcal{M} is contained in Z .

- there is a $\mathcal{A}'_\mu \boxtimes \mathcal{A}'_{\mu'}$ -lattice $\mathcal{M}' \subset \mathcal{M}$ such that any global function vanishing on the stable diagonal kills the coherent sheaf $\mathcal{M}'/h\mathcal{M}'$. This is a condition which should be thought of as an analogue of regularity of D-modules.

Proposition 3.3 *The image of \mathcal{G}_λ lies in the 2-category $\mathbf{HC}^g(\lambda)$.*

Proof. Since $\mathbf{HC}^g(\lambda)$ is closed under convolution, we need only check these conditions for \mathcal{E}_i and \mathcal{F}_i . We have already checked that the supports of these modules are Hecke correspondences, and thus lie in Z .

Furthermore, the D-modules $\tilde{\mathcal{E}}_i$ and $\tilde{\mathcal{F}}_i$ are the pushforwards of regular D-modules, and thus themselves regular. The corresponding very good filtrations on these D-modules have associated graded killed by any global function which vanishes on their support. Note that global functions on $\mathfrak{M}_\mu^\lambda \times \mathfrak{M}_{\mu \pm \alpha_i}^\lambda$ are the same as invariant functions on $E_\mu^\lambda \times E_{\mu \pm \alpha_i}^\lambda$. Since any invariant function whose reduction vanishes on the stable diagonal must vanish on the support of $\tilde{\mathcal{E}}_i$ and $\tilde{\mathcal{F}}_i$, it acts trivially on their associated graded. Thus, it also acts trivially on the induced lattice on \mathcal{E}_i or \mathcal{F}_i , and we are done. \square

This observation is useful, since it means that the modules supported on any system of subvarieties \mathfrak{M}_μ^λ which is closed under convolution with Z is closed under this categorical action. Examples include:

- the **cores** of the varieties \mathfrak{M}_μ^λ ; that is, the subvariety of representations which are nilpotent as representations of the preprojective algebra.
- the points attracted to the core under a \mathbb{C}^* -action for which the symplectic form has positive weight. The modules supported on these subvarieties (subject to a regularity condition like \mathbf{HC}^g) are an analogue of category \mathcal{O} and will be studied in much greater detail by Braden, Licata, Proudfoot and the author in forthcoming work [BLPW].

The question of the Grothendieck group of the modules supported on the core is a very interesting one. Examples show a subtle dependence on the choice of parameter, and we believe the techniques of this paper can shed significant light on the structure of this category.

This draws an analogy between the categorical action \mathcal{G}_λ , and the action of the monoidal category of Harish-Chandra bimodules (in the classical sense) on various categories of representations of \mathfrak{g} . The latter is a categorification of the Hecke algebra, which has

- its original representation-theoretic description,
 - a geometric one via the localization theorem of Beilinson and Bernstein [BB81],
- and

- a diagrammatic description in the guise of Soergel bimodules given by the work of Elias and Khovanov in type A [EK] and forthcoming work of Elias and Williamson in general [EW].

The 2-category \mathcal{U} was first defined in a purely diagrammatic manner, so it is striking evidence of its naturality (at least to the author) to see it arise in a geometric context as well.

There is also a purely representation-theoretic categorical action. We are interested in the \mathbb{C}^* -invariant section algebra $A_\mu = \Gamma(\mathfrak{M}_\mu^\lambda, \mathcal{A}_\mu)^{\mathbb{C}^*}$; this is a filtered algebra whose associated graded is the function algebra of \mathfrak{M}_μ^λ . As is shown by [BPW, 6.6], taking \mathbb{C}^* -invariant derived sections is a natural transformation from \mathcal{Q}^λ to the 2-category where

- objects are dimension vectors,
- 1-morphisms are objects in the bounded-below derived category of bimodules over A_μ and $A_{\mu'}$ with composition given by derived tensor product,
- 2-morphisms are just morphisms in the bounded-below derived category of bimodules over A_μ and $A_{\mu'}$.

Of course, it's most natural to think of this 2-category sitting inside functors acting on the derived categories of modules over the different A_μ ; this inclusion is given by taking derived tensor product.

Corollary 3.4 *The composition of \mathcal{G}_λ with $\Gamma(\mathfrak{M}_\mu^\lambda, -)^{\mathbb{C}^*}$ defines a categorical \mathfrak{g} -action on the derived categories of modules over the rings A_μ . This categorical action preserves the category of finite-dimensional modules.*

Finally, we turn to understanding how this action decategorifies. As defined in [BPW, §6.2], based on work of Kashiwara and Schapira [KS], we have a map CC from the K -group of sheaves supported on Z to $H_{top}^{BM}(Z)$ which intertwines convolution of sheaves with convolution of Borel-Moore classes. Composing the map induced on Grothendieck groups defined by \mathcal{G}_λ with CC, we obtain a homomorphism $C : K(\mathcal{U}) \rightarrow H_{top}^{BM}(Z)$.

Proposition 3.5 *We have a commutative diagram*

$$\begin{array}{ccc} & \dot{\mathcal{U}} & \\ \sim \nearrow & & \searrow N \\ K(\mathcal{U}) & \xrightarrow{C} & H_{top}^{BM}(Z) \end{array}$$

where $N : \dot{\mathcal{U}} \rightarrow H_{top}^{BM}(Z)$ is exactly the map defined by Nakajima in [Nak98].

Proof. The map CC sends $[\mathcal{E}_i]$ to the sum of the fundamental classes of the components of its support variety, weighted by the generic dimension of the stalk of its classical limit at a generic point of the component. However, the map $\tilde{X}_{\mu, \alpha_i}^\lambda \rightarrow \tilde{X}_\mu^\lambda \times \tilde{X}_{\mu - \alpha_i}^\lambda$ is injective with smooth image; it is locally modeled on the map from the $(k, k+1)$ -type partial flag variety to the 2 Grassmannians (with the ambient space given by the sum of all the other spaces, weighted by the number of arrows from i to that vertex). Thus, its pushforward has irreducible characteristic variety with multiplicity one.

The intersection of this characteristic variety with the stable locus is the support variety of \mathcal{E}_i , which we can thus identify with the Hecke correspondence denoted \mathfrak{P}_i in [Nak98]. By a symmetric argument, the support variety of \mathcal{F}_i is the variety obtained from this one by reversing factors. Thus, we have that

$$[\mathcal{E}_i] \mapsto [\mathcal{E}_i] \mapsto [\mathfrak{P}_i] \quad [\mathcal{F}_i] \mapsto [\mathcal{F}_i] \mapsto [\omega(\mathfrak{P}_i)].$$

By [Nak98, 9.4], the homomorphism $N: \dot{U} \rightarrow H_{top}^{BM}(Z)$ is the unique one with this property. \square

4. THE PROOF OF THEOREM 3.1

Now we proceed to the proof of Theorem 3.1 through a series of lemmata.

Lemma 4.1 *We have a natural isomorphism $M \star rN \cong r(r_*M \star N)$ for any complex of $\mathcal{A}_\mu \boxtimes \mathcal{A}_{\mu'}^{op}$ -modules M and any complex of $\mathcal{D}_{X_{\mu'}^\lambda} \boxtimes \mathcal{D}_{X_{\mu''}^\lambda}^{op}$ -modules N .*

Proof. We need only confirm this isomorphism when $M = M_1 \boxtimes M_2$ and $N = \mathcal{D}_{X_{\mu'}^\lambda} \boxtimes \mathcal{D}_{X_{\mu''}^\lambda}$. In this case,

$$M \star rN \cong M_1 \boxtimes \Gamma(\mathfrak{M}_{\mu'}^\lambda; M_2) \boxtimes \mathcal{A}_{\mu''} \quad r(r_*M \star N) \cong M_1 \boxtimes \Gamma(X_{\mu'}^\lambda; r_*M_2) \boxtimes \mathcal{A}_{\mu''}$$

The result then follows from the isomorphism

$$\Gamma(\mathfrak{M}_{\mu'}^\lambda; M_2) \cong \text{Hom}_{\mathcal{A}_{\mu'}}(\mathcal{A}_{\mu'}, M_1) \cong \text{Hom}_{\mathcal{D}_{X_{\mu'}^\lambda}}(\mathcal{D}_{X_{\mu'}^\lambda}, r_*M_2) \cong \Gamma(X_{\mu'}^\lambda; r_*M_2). \quad \square$$

Lemma 4.2 $r(\tilde{\mathcal{F}}_{i_1} \star \cdots \star \tilde{\mathcal{F}}_{i_n}) \cong \mathcal{F}_{i_1} \star \cdots \star \mathcal{F}_{i_n}$

Proof. We induct on n ; when $n = 1$, this is true by definition.

By the inductive hypothesis, $r(\tilde{\mathcal{F}}_{i_1} \star \cdots \star \tilde{\mathcal{F}}_{i_{n-1}}) \cong \mathcal{F}_{i_1} \star \cdots \star \mathcal{F}_{i_{n-1}}$. Thus, we have maps

$$r!(\mathcal{F}_{i_1} \star \cdots \star \mathcal{F}_{i_{n-1}}) \rightarrow \tilde{\mathcal{F}}_{i_1} \star \cdots \star \tilde{\mathcal{F}}_{i_{n-1}} \rightarrow r_*(\mathcal{F}_{i_1} \star \cdots \star \mathcal{F}_{i_{n-1}})$$

which induce isomorphisms after applying r . If we apply Lemma 4.1 with $M = \mathcal{F}_{i_1} \star \cdots \star \mathcal{F}_{i_{n-1}}$ and $N = \tilde{\mathcal{F}}_{i_n}$, we arrive at

$$\mathcal{F}_{i_1} \star \cdots \star \mathcal{F}_{i_n} \cong r(r!(\mathcal{F}_{i_1} \star \cdots \star \mathcal{F}_{i_{n-1}}) \star \tilde{\mathcal{F}}_{i_n}).$$

We have a map $a: r_!(\mathcal{F}_{i_1} \star \cdots \star \mathcal{F}_{i_{n-1}}) \rightarrow \tilde{\mathcal{F}}_{i_1} \star \cdots \star \tilde{\mathcal{F}}_{i_{n-1}}$ which induces an isomorphism after applying r . Thus $C(a)$, the cone of this morphism, has cohomology microsupported on the unstable locus. We have an exact triangle

$$r_!(\mathcal{F}_{i_1} \star \cdots \star \mathcal{F}_{i_{n-1}}) \star \tilde{\mathcal{F}}_{i_n} \rightarrow \tilde{\mathcal{F}}_{i_1} \star \cdots \star \tilde{\mathcal{F}}_{i_n} \rightarrow C(a) \star \tilde{\mathcal{F}}_{i_n} \xrightarrow{[1]}$$

since $- \star \tilde{\mathcal{F}}_{i_n}$ is a triangulated functor. As usual, the microsupport of $\tilde{\mathcal{F}}_i$ lies in the image in $T^*X_\mu^\lambda \times T^*X_{\mu-\alpha_i}^\lambda$ of

$$\{x \in X_{\mu;\alpha_i}^\lambda, \varphi \in T_{(p_1(x), p_2(x))}^*(X_\mu^\lambda \times X_{\mu-\alpha_i}^\lambda) \mid (p_1 \times p_2)^* \varphi = 0\}.$$

We can think of x as a representation of the oriented quiver with chosen subrepresentation, the covector φ as a choice of maps along the oppositely oriented arrows that extend the total representation and the subrepresentation to representations of the preprojective algebra; in this case, the vanishing condition is simply that the inclusion is a map of preprojective representations. In particular, if the lefthand point $p_1(x)$ has a destabilizing subrepresentation, $p_2(x)$ does as well. That is, the microsupport of $\tilde{\mathcal{F}}_i$ has the property that if $p_1(x)$ is unstable, then $p_2(x)$ is as well. So the property of having unstable support is preserved by $- \star \tilde{\mathcal{F}}_{i_n}$.

In particular, the support of $C(a) \star \tilde{\mathcal{F}}_{i_n}$ lies in the unstable locus, so $r(C(a) \star \tilde{\mathcal{F}}_{i_n}) = 0$. Thus, we arrive at the desired isomorphism

$$\mathcal{F}_{i_1} \star \cdots \star \mathcal{F}_{i_n} \cong r(r_!(\mathcal{F}_{i_1} \star \cdots \star \mathcal{F}_{i_{n-1}}) \star \tilde{\mathcal{F}}_{i_n}) \cong r(\tilde{\mathcal{F}}_{i_1} \star \cdots \star \tilde{\mathcal{F}}_{i_n}). \quad \square$$

Proof of Theorem 3.1. We wish to show that this action is a **Q-strong g-action** in the sense of [CL, 1.2]. This is defined by a list of conditions, which we check with numbering as in [CL]

- (1) the integrability follows from the fact that for fixed μ , there are only finitely many integers such that $\mathfrak{M}_{\mu+k\alpha_i}^\lambda$ is non-empty.
- (2) $\text{Mor}_{\mathbb{Q}^\lambda}(\text{id}_\mu, \text{id}_\mu) \cong \text{Ext}_{\mathcal{A}_\mu \boxtimes \mathcal{A}_\mu^{\text{op}}}^\bullet(\mathcal{A}_\mu, \mathcal{A}_\mu)$ is the same as the Hochschild cohomology of the algebra \mathcal{A}_μ , which is obviously positively graded.

The remaining conditions all follow immediately from calculations done on the level of constructible sheaves carried through the Riemann-Hilbert correspondence. Condition (4) is an immediate consequence of work of Vasserot and Varagnolo [VV11]. Conditions (3) and (5) simply state that certain isomorphisms exist; these are shown in the characteristic p setting by [Lib, 1.12-13]. Li's proofs use no special facts about characteristic p fields; in principle, we could simply cite his work, but for the same of completeness, we give arguments in the deformation quantization setting for the same facts.

- (5) We have an isomorphism of $\tilde{\mathcal{F}}_i \star \tilde{\mathcal{E}}_j$ with the pushforward of the structure sheaf of $X_{\mu;\alpha_i}^\lambda \times_{X_{\mu-\alpha_i}^\lambda} X_{\mu-\alpha_i+\alpha_j;\alpha_j}^\lambda$ tensored with the canonical sheaf of the right factor. Similarly, $\tilde{\mathcal{E}}_j \star \tilde{\mathcal{F}}_i$ is isomorphic to the analogous pushforward from $X_{\mu+\alpha_j;\alpha_j}^\lambda \times_{X_{\mu+\alpha_j}^\lambda} X_{\mu+\alpha_j;\alpha_i}^\lambda$. The first variety consists can be thought of as a triple

of representations, with the outer two realized as subrepresentations of the center; the second of a triple where the middle representation is a subrepresentation of the right and the left (with all inclusions codimension 1). Since the right and left representations have different dimension vectors, they cannot coincide, and we have an isomorphism

$$X_{\mu;\alpha_i}^\lambda \times_{X_{\mu-\alpha_i}^\lambda} X_{\mu-\alpha_i+\alpha_j;\alpha_j}^\lambda \rightarrow X_{\mu+\alpha_j;\alpha_j}^\lambda \times_{X_{\mu+\alpha_j}^\lambda} X_{\mu+\alpha_j;\alpha_i}^\lambda$$

forgetting the middle representation and replacing it with the intersection of the left and right. Thus $\tilde{\mathcal{F}}_i \star \tilde{\mathcal{E}}_j \cong \tilde{\mathcal{E}}_j \star \tilde{\mathcal{F}}_i$.

We wish to upgrade this to an isomorphism after reduction; the reduced versions $\mathcal{F}_i \star \mathcal{E}_j$ and $\mathcal{E}_j \star \mathcal{F}_i$ are obtained by microlocalizing the corresponding D-modules on $X_{\mu;\alpha_i}^\lambda \times_{X_{\mu-\alpha_i}^\lambda} X_{\mu-\alpha_i+\alpha_j;\alpha_j}^\lambda$ and $X_{\mu+\alpha_j;\alpha_j}^\lambda \times_{X_{\mu+\alpha_j}^\lambda} X_{\mu+\alpha_j;\alpha_i}^\lambda$ and applying pushforward after restricting to the stable loci. The difference between convolution before reduction or after reduction is whether we impose the stability condition on the middle representation in our description as triples. However, in either description requiring stability on the right and left assures in the middle; this is obvious for a common subrepresentation, but for a common superrepresentation, the destabilizing subrepresentation must non-trivially intersect the left or right. Thus, we have the desired isomorphism

$$\mathcal{F}_i \star \mathcal{E}_j \cong \mathcal{E}_j \star \mathcal{F}_i.$$

- (3) Fix a node i . Applying Fourier transform as necessary, we can assume that i is a source. Proposition 3.2 assures us that we can transition between the different quantizations that arise from different orientations; alternatively we can use the constructions of quantizations of line bundles to bimodules given in [BPW, 5.2]. We let \hat{X}_μ^λ be the open locus in X_μ^λ where the sum x_{out} of the maps along edges pointing out from i is injective; and let $\tilde{X}_{\mu\pm\alpha_i;\alpha_i}^\lambda$ be the restriction of the appropriate correspondence to the same locus. We have the following diagram of morphisms

$$\begin{array}{ccccc} & & \tilde{X}_{\mu-\alpha_i;\alpha_i}^\lambda & & \tilde{X}_{\mu;\alpha_i}^\lambda & & \tilde{X}_{\mu+\alpha_i}^\lambda \\ & \swarrow f_1 & & \searrow f_2 & \swarrow e_1 & & \searrow e_2 \\ \tilde{X}_{\mu-\alpha_i}^\lambda & & & & \tilde{X}_\mu^\lambda & & \tilde{X}_{\mu+\alpha_i}^\lambda \\ \downarrow \iota_- & & & & \downarrow \iota & & \downarrow \iota_+ \\ X_{\mu-\alpha_i}^\lambda & & & & X_\mu^\lambda & & X_{\mu+\alpha_i}^\lambda \end{array}$$

Note that the maps e_i and f_i are both proper and smooth; their fibers are projective spaces. We let $\hat{\mathcal{F}}_i$ denote the restriction of $\tilde{\mathcal{F}}_i$ to the injective locus, and similarly for $\hat{\mathcal{E}}_i$.

We have an isomorphism of $\hat{\mathcal{F}}_i \star \hat{\mathcal{E}}_i$ with the pushforward of the structure sheaf of $\hat{X}_{\mu;\alpha_i}^\lambda \times_{\hat{X}_{\mu-\alpha_i}^\lambda} \hat{X}_{\mu;\alpha_i}^\lambda$ tensored with the canonical sheaf of the right factor. The sheaf $\hat{\mathcal{E}}_i \star \hat{\mathcal{F}}_i$ is derived in the same way from $\hat{X}_{\mu+\alpha_i;\alpha_i}^\lambda \times_{\hat{X}_{\mu+\alpha_i}^\lambda} \hat{X}_{\mu+\alpha_i;\alpha_i}^\lambda$.

By the same argument as in condition (5), these spaces can be interpreted as triples of representations, now with the left and right having the same dimension vector, either with the middle containing the left and right, or being a submodule in both. Also, the same argument shows that in this case convolution commutes with reduction; if the middle representation is unstable, the destabilizing representation isn't just supported on the vertex i by the injectivity condition, and thus it non-trivially intersects the left and right representations, destabilizing them.

Thus, we have

$$\mathcal{F}_i \star \mathcal{E}_j \cong r(\hat{\mathcal{F}}_i \star \hat{\mathcal{E}}_i) \quad \mathcal{E}_j \star \mathcal{F}_i \cong r(\hat{\mathcal{E}}_j \star \hat{\mathcal{F}}_i).$$

So, we wish to show that

$$\begin{aligned} \hat{\mathcal{F}}_i \star \hat{\mathcal{E}}_i &\cong \hat{\mathcal{E}}_j \star \hat{\mathcal{F}}_i \oplus (q^{\langle \alpha_i, \mu \rangle + d_i} + \dots + q^{-\langle \alpha_i, \mu \rangle - d_i}) \cdot \mathcal{D}_\Delta \quad \text{if } \langle \alpha_i, \mu \rangle \leq 0 \\ \hat{\mathcal{F}}_i \star \hat{\mathcal{E}}_i &\oplus (q^{-\langle \alpha_i, \mu \rangle + d_i} + \dots + q^{\langle \alpha_i, \mu \rangle - d_i}) \cdot \mathcal{D}_\Delta \cong \hat{\mathcal{E}}_j \star \hat{\mathcal{F}}_i \quad \text{if } \langle \alpha_i, \mu \rangle \geq 0 \end{aligned}$$

where Δ denotes the diagonal in $\hat{X}_\mu^\lambda \times \hat{X}_\mu^\lambda$. Since this is a statement about an isomorphism of D-modules, it suffices to prove the corresponding isomorphism of sheaves on the other side of Riemann-Hilbert. Since this is a statement about isomorphisms between sums of pushforwards along maps between varieties defined over the integers, it suffices to prove the corresponding statement after base change to an algebraically closed field of characteristic p with ℓ -adic sheaves rather than analytic ones. This is done by Li in [Lib, 1.13]; specifically, his equations [Lib, (19-20)] compute the two sides of the proceeding displayed equations and show that they agree. As we noted earlier, this difference in characteristics of base fields is one of convention, and not of any geometric significance. Thus, applying the functor r to both equations, we have arrived at the desired isomorphisms:

$$\begin{aligned} \mathcal{F}_i \star \mathcal{E}_i &\cong \mathcal{E}_j \star \mathcal{F}_i \oplus (q^{\langle \alpha_i, \mu \rangle + d_i} + \dots + q^{-\langle \alpha_i, \mu \rangle - d_i}) \cdot \mathcal{A}_\mu \quad \text{if } \langle \alpha_i, \mu \rangle \leq 0 \\ \mathcal{F}_i \star \mathcal{E}_i &\oplus (q^{-\langle \alpha_i, \mu \rangle + d_i} + \dots + q^{\langle \alpha_i, \mu \rangle - d_i}) \cdot \mathcal{A}_\mu \cong \mathcal{E}_j \star \mathcal{F}_i \quad \text{if } \langle \alpha_i, \mu \rangle \geq 0. \end{aligned}$$

- (4) This condition states that the sheaves $\oplus_i \mathcal{F}_{i_1} \star \dots \star \mathcal{F}_{i_n}$ carry an action of the KLR algebra for the polynomials Q we have specified. The solution sheaf (i.e. the image under the Riemann-Hilbert correspondence) of $\hat{\mathcal{F}}_{i_1} \star \dots \star \hat{\mathcal{F}}_{i_n}$ is precisely the perverse sheaf that Varagnolo and Vasserot denote by ${}^\delta \mathcal{L}_i$ in [VV11].

Thus, in our language, [VV11, 3.5] shows that the Ext algebra of solution sheaves of $\oplus_i \tilde{\mathcal{F}}_{i_1} \star \cdots \star \tilde{\mathcal{F}}_{i_n}$ is given by the KLR algebra $R = \oplus R_v$; since the Riemann-Hilbert correspondence is an equivalence of categories, we arrive at an isomorphism

$$\mathrm{Ext}^\bullet \left(\oplus_i \tilde{\mathcal{F}}_{i_1} \star \cdots \star \tilde{\mathcal{F}}_{i_n} \right) \cong R.$$

It follows that the image of these sheaves under any functor, in particular $r(\tilde{\mathcal{F}}_{i_1} \star \cdots \star \tilde{\mathcal{F}}_{i_n}) \cong \mathcal{F}_{i_1} \star \cdots \star \mathcal{F}_{i_n}$, still carry this action.

Thus, by [CL, Theorem 1.1], we, in fact, have a functor from \mathcal{U} to \mathcal{Q}^λ as desired. \square

This completes the proof of Theorem A.

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